MATH 147, SPRING 2021: SOLUTIONS TO PRACTICE PROBLEMS FOR EXAM 3

Practice Problems.

1. Calculate $\int\int\int_B y^2z^2\ dV$ for B the solid bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.
Solution. If we let D denote the unit disk in the yz-plane, then

$$\begin{aligned}
\int\int\int_B y^2z^2\ dV &= \int\int_D \int_0^{1-y^2-z^2} y^2z^2\ dx\ dA \\
&= \int\int_D (1-y^2-z^2)y^2z^2\ dA \\
&= \int_0^{2\pi}\int_0^1 (1-r^2)(r\cos(\theta))^2(r\sin(\theta))^2\ rdrd\theta \\
&= \int_0^{2\pi}\int_0^1 (r^5-r^7)\cos^2(\theta)\sin^2(\theta)\ drd\theta \\
&= (\frac{1}{6}-\frac{1}{8})\int_0^{2\pi}\cos^2(\theta)\sin^2(\theta)\ d\theta \\
&= \frac{1}{24}\int_0^{2\pi}\frac{1}{8}-\frac{1}{8}\cos(4\theta)\ d\theta \ \text{(double angle formula twice)} \\
&= \frac{1}{24}\cdot\{\frac{\theta}{8}-\frac{1}{32}\sin(4\theta)\}_0^{2\pi} \\
&= \frac{1}{24}\cdot\frac{2\pi}{8} \\
&= \frac{\pi}{96}.
\end{aligned}$$

2. Calculate $\int\int\int_B z^3\sqrt{x^2+y^2+z^2}\ dV$, for B the solid hemisphere with radius 1 and $z \geq 0$.
Solution. Using spherical coordinates,

$$\begin{aligned}
\int\int\int_B z^3\sqrt{x^2+y^2+z^2}\ dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\cos(\phi))^3\sqrt{\rho^2}\rho^2\sin(\phi)\ d\rho\phi d\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}}\int_0^1 \rho^6\cos^3(\phi)\sin(\phi)\ d\rho d\phi \\
&= \frac{2\pi}{7}\int_0^{\frac{\pi}{2}}\cos^3(\phi)\sin(\phi)\ d\phi \\
&= \frac{2\pi}{7}\cdot(-\frac{\cos^4(\theta)}{4})\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{14}.
\end{aligned}$$

3. Let B_0 be the parallelepiped spanned by the vectors $2\vec{i} - \vec{j} + k, 3\vec{i} + k, 4\vec{j} - \vec{k}$ and B the parallelepiped obtained by translating the corner of B_0 at the origin to the point (3,2,1). Calculate $\int\int\int_B 2x - y + 3z\ dV$.

Solution. We first write the transformation $G(u, v, w)$ that takes the unit cube in uvw-space to B_0. For a reminder on how to do this, see the lecture of Thursday, April 8. $G(u, v, w) = (2u + 3v, -u + 4w, u + v - w)$, with $(u, v, w) \in [0, 1] \times [0, 1] \times [0, 1]$. Now to translate B_0 to B, we just add $(3, 2, 1)$ to the coordinates of $G(u, v, w)$ to get a the transformation $H(u, v, w) = (2u + 3v + 3, -u + 4w + 2, u + v - w + 1)$, which takes the unit cube in uvw-space to B. Taking the Jacobian of $H(u, v, w)$, we get

$$\text{Jac}(H) = \det \begin{pmatrix} 2 & -1 & 1 \\ 3 & 0 & 1 \\ 0 & 4 & -1 \end{pmatrix} = 1.$$

Thus,

$$\begin{aligned}
\int\int\int_B 2x - y + 3z \; dv &= \int_0^1 \int_0^1 \int_0^1 \{2(2u + 3v + 3) - (-u + 4w + 2) + 3(u + v - w + 1)\} \cdot 1 \; dV \\
&= \int_0^1 \int_0^1 \int_0^1 8u + 9v - 7w + 7 \; du \; dv \; dw \\
&= \int_0^1 \int_0^1 11 + 9v - 7w \; dv \; dw \\
&= \int_0^1 \frac{31}{2} - 7w \; dw \\
&= 12.
\end{aligned}$$

4. Let C be a curve that starts at the north pole of the sphere of radius R centered at the origin and ends at the south pole. You can choose any such C. Find a parametrization of C and calculate the line integrals $\int_C x + y + z \; ds$ and $\int_C \mathbf{F} \cdot d\mathbf{r}$, for $\mathbf{F} = xi + yj + zk$.

Solution. We can take the curve $\mathbf{r}(t) = (R\sin(t), 0, R\cos(t))$, $0 \leq t \leq \pi$, though fixing θ at any angle and letting ϕ run from 0 to π will work. Note $\mathbf{r}'(t) = (R\cos(t), 0, -R\sin(t))$, $||\mathbf{r}'(t)|| = R$.

$$\begin{aligned}
\int_C x + y + z \; ds &= \int_0^\pi (R\sin(t) + 0 + R\cos(t)) \; R \; dt \\
&= R^2 \cdot (-\cos(t) + \sin(t))\Big|_0^\pi \\
&= 2R^2.
\end{aligned}$$

For the line integral of $\mathbf{F}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (R\sin(t), 0, R\cos(t)) \cdot (R\cos(t), 0, -R\sin(t)) = 0$, so $\int_C \mathbf{F} \cdot \mathbf{r}(t) = 0$.

5. Let S denote the top half of the sphere of radius R centered at the origin and consider the scalar function $f(x, y, z) = \sqrt{1 + x^2 + y^2 + z^2}$. Calculate $\int\int_S f(x, y, z) \; dS$ in two ways :

(i) Intuitively, by just writing down your answer. You must justify this.
(ii) Directly, using a parametrization of S.

Solution. (i) Answer: $2\pi R^2 \sqrt{1 + R^2}$. Reason: $f(x, y, z)$ takes the constant value $\sqrt{1 + R^2}$ on S, so that $\int\int_S f(x, y, z) \; dS$ is just $\sqrt{1 + R^2}$ times the surface area of S, which is $2\pi R^2$, half of the surface area of the sphere of radius R.

(ii) Using the standard spherical parametrization, where $||\mathbf{T}_\phi \times \mathbf{T}_\theta|| = R^2\sin(\phi)$,

$$\begin{aligned}\int\int_S \sqrt{1+x^2+y^2+z^2}\ dS &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sqrt{1+R^2}\ R^2\sin(\phi)d\phi d\theta \\ &= 2\pi\sqrt{1+R^2}R^2\int_0^{\frac{\pi}{2}} \sin(\phi)\ d\phi \\ &= 2\pi R^2\sqrt{1+R^2}(-\cos(\phi))\Big|_0^{\frac{\pi}{2}} \\ &= 2\pi R^2\sqrt{1+R^2}.\end{aligned}$$

6. For the vector field $\mathbf{F} = 3z^2\vec{k}$, use the limit definition of the divergence with decreasing spheres to show that div $\mathbf{F}(P) = 6z_0$, for any point $P = (x_0, y_0, z_0)$.

Solution. Let S_ϵ denote the sphere of radius ϵ centered at P. BY definition, w have,

$$\text{div } \mathbf{F}(P) = \lim_{\epsilon\to 0}\frac{1}{\text{vol}(S_\epsilon)}\int\int_{S_\epsilon}\mathbf{F}\cdot\mathbf{n}\ dS = \lim_{\epsilon\to 0}\frac{1}{\text{vol}(S_\epsilon)}\int\int_{S_\epsilon}\mathbf{F}\cdot d\mathbf{S}.$$

Since S_ϵ is just the sphere of radius R centered at the origin, translated to the point P, we may parametrize S_ϵ as follows:

$$G(\phi,\theta) = (\epsilon\sin(\phi)\cos(\theta) + x_0, \epsilon\sin(\phi)\sin(\theta) + y_0, \epsilon\cos(\phi) + z_0).$$

Thus, $\mathbf{F}(G(u,v)) = 3(\epsilon\cos(\phi) + z_0)^2\vec{k}$. It is easy to check that

$$\mathbf{T}_u \times \mathbf{T}_v = \epsilon\sin^2(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)),$$

so that

$$\mathbf{F}(G(u,v))\cdot\mathbf{T}_u\times\mathbf{T}_v = 3(\cos(\phi)+z_0)^2\cdot\epsilon^2\sin(\phi)\cos(\phi).$$

Therefore,

$$\begin{aligned}\int\int_{S_\epsilon}\mathbf{F}\cdot d\mathbf{S} &= \int\int_{S_\epsilon}\mathbf{F}(G(u,v))\cdot\mathbf{T}_u\times\mathbf{T}_v\ dA \\ &= \int_0^{2\pi}\int_0^{\pi} 3(\epsilon\cos(\phi)+z_0)^2\cdot\epsilon^2\sin(\phi)\cos(\phi)\ d\phi\ d\theta \\ &= 3\epsilon^2\int_0^{2\pi}\int_0^{\pi}(\epsilon\cos(\phi)+z_0)^2\cdot\sin(\phi)\cos(\phi)\ d\phi\ d\theta \\ &= 6\pi\epsilon^2\int_0^{\pi}(\epsilon\cos(\phi)+z_0)^2\cdot\sin(\phi)\cos(\phi)\ d\phi \\ &= 6\pi\epsilon^2\int_0^{\pi}\epsilon^2\cos^3(\phi)\sin(\phi) + 2\epsilon\cos^2(\phi)\sin(\phi)z_0 + z_0^2\sin(\phi)\cos(\phi)\ d\phi \\ &= 6\pi\epsilon^2\{-\frac{1}{4}\epsilon^2\cos^4(\phi)\Big|_0^{\pi} - \frac{2}{3}\epsilon\cos^3(\phi)z_0\Big|_0^{\pi} + z_0\sin^2(\phi)\Big|_0^{\pi}\} \\ &= 6\pi\epsilon^2\{0 + \frac{4}{3}\pi\epsilon z_0 + 0\} \\ &= 8\pi\epsilon^3 z_0.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{\text{vol}(S_\epsilon)}\int\int_{S_\epsilon}\mathbf{F}\cdot\mathbf{n}\ dS &= \lim_{\epsilon\to 0}\frac{1}{\frac{4\pi\epsilon^3}{3}}\cdot 8\pi\epsilon^3 z_0 \\ &= \lim_{\epsilon\to 0} 6z_0 = 6z_0,\end{aligned}$$

as required.

7. Let S be that portion of the plane $x + y - z = 0$, with $0 \le x \le a$ and $0 \le y \le b$, for $a, b > 0$ and let $\mathbf{F} = x\vec{i} + 2y\vec{j} + 3\vec{k}$. Show by direct calculation that the surface integral $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ can change, depending upon the parametrization of S. Be sure to exhibit explicitly two different parametrizations of S.

Solution. Take the two parametrizations $G(u, v) = (u, v, u + v)$ with $0 \le u \le a, 0 \le v \le b$ and $H(u, v) = (v, u, u + v)$, with $0 \le v \le a, 0 \le u, \le b$. In the first case, $\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -i - j + k$ and in the second case, $\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = i + j - k$. Integrating $\mathbf{F}$ with the first parametrization gives $a^2b + \frac{ab^2}{2}$, while integrating with the second parametrization gives $-(a^2b + \frac{ab^2}{2})$.

8. Let S be that portion of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$.

(i) Calculate the surface area of S.

(ii) Calculate the surface integral $\int\int_S x^2 + y^2 \; dS$.

Solution. Take $G(u, v) = (u, v, u^2 + v^2)$, with $D : 0 \le u^2 + v^2 \le 4$, for the parametrization of S. Then

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1),$$

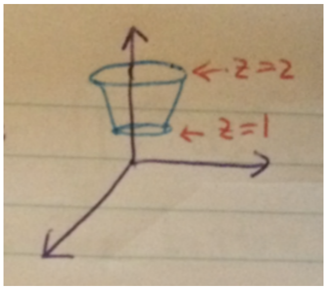
so $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{1 + 4u^2 + 4v^2}$. For the surface area we have:

$$\begin{aligned} \text{surface area}(S) &= \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v|| \; dudv \\ &= \int\int_D \sqrt{1 + 4u^2 + 4v^2} \; dA \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \; rdrd\theta \\ &= 2\pi \int_0^2 \sqrt{1 + 4r^2} \; rdr \\ &= \frac{2\pi}{12}(1 + 4r^2)^{\frac{3}{2}} \Big|_0^2 \\ &= \frac{\pi}{6}(17^{\frac{3}{2}} - 1). \end{aligned}$$

For the surface integral, we have that on S, $x^2 + y^2 = u^2 + v^2$, thus

$$\begin{aligned} \int\int_S x^2 + y^2 \; dS &= \int\int_D (u^2 + v^2)\sqrt{4u^2 + 4v^2 + 1} \; dA \\ &= \int_0^{2\pi} \int_0^2 r^2 \sqrt{4r^2 + 1} \cdot r \; dr \; d\theta \\ &= 2\pi \int_0^2 r^3 \sqrt{4r^2 + 1} \; dr \\ &= \frac{2\pi}{32} \int_1^{17} u^{\frac{3}{2}} - u^{\frac{1}{2}} \; du, \text{ using } u\text{-substitution with } u = 4r^2 + 1 \\ &= \frac{\pi}{16} \cdot \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_{u=1}^{u=17} \\ &= \frac{\pi}{16} \cdot \left\{\frac{2}{5}17^{\frac{5}{2}} - \frac{2}{3}17^{\frac{3}{2}} + \frac{4}{15}\right\}. \end{aligned}$$

9. Let B be the solid bounded by the closed surface S which is that portion of the cone $z = \sqrt{x^2+y^2}$, with $1 \le z \le 2$. verify the Divergence Theorem for $\mathbf{F} = xi + yj + z^2 k$.



Solution. We first calculate $\int\int\int_B \text{div } \mathbf{F}\ dV$. We have div $\mathbf{F} = 2+2z$. To integrate over B, we use cylindrical coordinates noting that on the cone itself, $z = \sqrt{x^2+y^2} = r$. Thus,

$$\begin{aligned}
\int\int\int_B \text{div } \mathbf{F}\ dv &= \int\int\int_B 2+2z\ dV \\
&= \int_1^2 \int_0^{2\pi} \int_0^z (2+2z)\ r\ dr\ d\theta\ dz \\
&= 2\pi \int_1^2 (2+2z)\cdot \frac{r^2}{2}\Big|_{r=0}^{r=z} dz \\
&= \pi \int_1^2 2z^2 + 2z^3\ dz \\
&= \pi \cdot (\frac{2z^3}{3} + \frac{2z^4}{4})_{z=1}^{z=2} \\
&= \pi \cdot \{(\frac{16}{3} + \frac{32}{4}) - (\frac{2}{3} + \frac{2}{4})\} \\
&= \frac{73}{6}\pi.
\end{aligned}$$

If we let S denote the closed surface bounding B, then S has three parts, S_1, the top, S_2, the bottom, and S_3 that portion of the given cone with $1 \le z \le 2$. For S_1, $\mathbf{n} = \vec{k}$, thus $\mathbf{F}\cdot\mathbf{n} = z^2$. On S_1, $z = 2$, so $\mathbf{F}\cdot\mathbf{n} = 4$ on S_1. Thus,

$$\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{S_1} 4\ dS = 4\cdot \text{area}(S_1) = 16\pi.$$

For S_2, $\mathbf{n} = -\vec{k}$, so $\mathbf{F}\cdot\mathbf{n} = -z^2$. On S_2, $z = 1$, so $\mathbf{F}\cdot\mathbf{n} = -1$ on S_2. Therefore,

$$\int\int_{S_2} \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{S_1} -1\ dS = -1\cdot \text{area}(S_2) = -\pi.$$

To parametrize S_3, we take $G(u,v) = (v\cos(u), v\sin(u), v)$, with $0 \le u \le 2\pi$ and $1 \le v \le 2$. Thus, $\mathbf{F}(G(u,v)) = \cos(u)\vec{i} + \sin(u)\vec{j} + v^2\vec{k}$. We then have

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin(u) & v\cos(u) & 0 \\ \cos(u) & \sin(u) & 1 \end{vmatrix} = v\cos(u)\vec{i} + v\sin(u)\vec{j} + -v\vec{k},\ \text{the outward normal.}$$

Thus,

$$\mathbf{F}(G(u,v))\cdot\mathbf{T}_u\times\mathbf{T}_v = v^2\cos^2(u) + v^2\sin^2(u) - v^3 = v^2 - v^3.$$

Therefore,

$$\int\int_{S_3} \mathbf{F}\cdot d\mathbf{S} = \int\int_{S_3} \mathbf{F}(G(u,v))\cdot \mathbf{T}_u \times \mathbf{T}_v \ dA$$
$$= \int_1^2 \int_0^{2\pi} v^2 - v^3 \ du \ dv$$
$$= 2\pi \int_1^2 v^2 - v^3 \ dV$$
$$= 2\pi \cdot \{\frac{v^3}{3} - \frac{v^4}{4}\}_1^2$$
$$= -\frac{17}{6}\pi.$$

Adding the three surface integrals we have

$$\int\int_S \mathbf{F}\cdot\mathbf{n} \ dS = 16\pi - \pi - \frac{17}{6}\pi = \frac{73}{6}\pi,$$

as required.

10. Let B denote the solid sphere of radius R centered at the origin, and let $P = (0,0,R)$ denote the north pole. Find the average value of the distance of points $(x,y,z) \in B$ to the point P.

Solution. We need to find the average value of the function $f(x,y,z) = \sqrt{x^2+y^2+(z-R)^2}$ over the domain B. So we first calculate

$$\int\int\int_B \sqrt{x^2+y^2+(z-R)^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \sqrt{(\rho\sin(\phi)\cos(\theta))^2 + (\rho\sin(\phi)\sin(\theta))^2 + (\rho\cos(\phi)-R)^2} \ \rho^2\sin(\phi) \ d\rho \ d\phi \ d\theta$$
$$= 2\pi \int_0^{\pi}\int_0^{R} \sqrt{\rho^2+R^2-2\rho R\cos(\phi)}\rho^2\sin(\phi) \ d\rho \ d\phi$$
$$= 2\pi \int_0^{R}\int_0^{\pi} \sqrt{\rho^2+R^2-2\rho R\cos(\phi)}\rho^2\sin(\phi) \ d\phi \ d\rho.$$

We can use u-substitution on the inner integral, by setting $u = \rho^2 + R^2 - 2\rho R\cos(\phi)$. Then upon differentiating, $du = 2\rho R\sin(\phi) \ d\phi$, so that $\sin(\phi) \ d\phi = \frac{1}{2\rho R}du$. When $\phi = 0$, $u = (R-\rho)^2$ and when $\phi = \pi$, $u = (R+\rho)^2$, so continuing, we have

$$\int\int\int_B \sqrt{x^2+y^2+(z-R)^2} = \int_0^R \int_{(R-\rho)^2}^{(R+\rho)^2} \sqrt{u}\rho^2 \cdot \frac{1}{2\rho R} du \ d\rho$$
$$= \frac{\pi}{R}\int_0^R \int_{(R-\rho)^2}^{(R+\rho)^2} \rho\sqrt{u} \ du \ d\rho$$
$$= \frac{\pi}{R}\int_0^R \frac{2}{3}u^{\frac{3}{2}}\rho\Big|_{u=(R-\rho)^2}^{u=(R+\rho)^2} d\rho$$
$$= \frac{2\pi}{3R}\int_0^R \rho\{(R+\rho)^3 - (R-\rho)^3\} \ d\rho$$
$$= \frac{2\pi}{3R}\int_0^R 6R^2\rho^2 + 2\rho^4 \ d\rho$$
$$= \frac{2\pi}{3R}\{2R^2\rho^3 + \frac{2}{5}\rho^5\}_{\rho=0}^{\rho=R}$$
$$= \frac{8}{5}\pi R^4.$$

Thus,

$$\begin{aligned}\text{average distance to } (0,0,R) &= \frac{1}{\text{vol}(B)} \int\int\int_B \sqrt{x^2+y^2+(z-R)^2} \\ &= \frac{3}{4\pi R^3} \cdot \frac{8}{5}\pi R^4 \\ &= \frac{6R}{5}.\end{aligned}$$

11. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = x^2yi + y^2zj + z^2xk$, for $C : \mathbf{r}(t) = e^{-t}i + e^{-2t}j + e^{-3t}k$, with $0 \le t < \infty$.
Solution. We let C_b denote the curve given by $\mathbf{r}(t)$ with $0 < t < b$ and consider $\lim_{b\to 0} \int_{C_b} \mathbf{F} \cdot d\mathbf{r}$. If this limit exists, it will equal $\int_C \mathbf{F} \cdot d\mathbf{r}$.
We have $\mathbf{F}(\mathbf{r}(t)) = e^{-4t}\vec{i} + e^{-7t}\vec{j} + e^{-7t}\vec{k}$ and $\mathbf{r}'(t) = -e^{-t}\vec{i} - 2e^{-2t}\vec{k} - 3e^{-3t}\vec{k}$, so

$$\begin{aligned}\int_{C_b} \mathbf{F} \cdot d\mathbf{r} &= \int_0^b (e^{-4t}\vec{i} + e^{-7t}\vec{j} + e^{-7t}\vec{k}) \cdot (-e^{-t}\vec{i} - 2e^{-2t}\vec{k} - 3e^{-3t}\vec{k})\ dt \\ &= \int_0^b -e^{-5t} - 2e^{-9t} - 3e^{-10t}\ dt \\ &= (\frac{1}{5}e^{-5b} + \frac{2}{9}e^{-9b} + \frac{3}{10}e^{-10b}) - (\frac{1}{5} + \frac{2}{9} + \frac{3}{10}).\end{aligned}$$

It follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \lim_{b\to 0} \int_{C_b} \mathbf{F} \cdot d\mathbf{r} \\ &= \lim_{b\to 0}\{(\frac{1}{5}e^{-5b} + \frac{2}{9}e^{-9b} + \frac{3}{10}e^{-10b}) - (\frac{1}{5} + \frac{2}{9} + \frac{3}{10})\} \\ &= 0 - (\frac{1}{5} + \frac{2}{9} + \frac{3}{10}) \\ &= -\frac{13}{18}.\end{aligned}$$

12. Find the volume of the solid W, whose boundary is a closed surface, if

$$\int\int_{\partial W} \{(x+xy+z)\vec{i} + (x+3y-\frac{1}{2}y^2)\vec{j} + 4z\vec{k}\} \cdot d\mathbf{S} = 16.$$

Solution. If we set $\mathbf{F} = (x+xy+z)\vec{i} + (x+3y-\frac{1}{2}y^2)\vec{j} + 4z\vec{k}$, then we are given $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 16$, where S is the closed boundary of W. By the Divergence Theorem we have,

$$\begin{aligned}16 &= \int\int_S \mathbf{F} \cdot d\mathbf{S} \\ &= \int\int\int_W \text{div } \mathbf{F}\ V \\ &= \int\int\int_W 1 + y + 3 - y + 4\ dV \\ &= \int\int\int_W 8\ dV \\ &= 8 \cdot \text{volume}(W).\end{aligned}$$

Thus, volume$(W) = 2$.